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LETTER TO THE EDITOR

Fractal dimension of intersection sets in the dielectric breakdown model

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Abstract. Clusters grown with the dielectric breakdown model (DBM) in the cylinder geometry show two growth phases: a scaling regime for cluster heights smaller than the cylinder circumference and a subsequent steady state, which is translational invariant in the main growth direction. The box-counting dimension of one-dimensional intersection sets of the clusters is studied for six different values of the growth parameter η and four cylinder circumferences. We find that in the scaling regime this dimension depends on the height at which the intersection is made and the clusters are thus inhomogeneous. The results also show that clusters in the steady state are translational invariants in the main growth direction and self-similar in the direction perpendicular to it. A comparison between our findings and the theoretical results obtained from the fixed scale transformation approach by Pietronero *et al*, shows good agreement.

The dielectric breakdown model (DBM) [1] is a growth model well known to give rise to fractal structures with fractal dimensions [2] depending on the growth parameter called η. For $\eta = 1$ one recovers the diffusion-limited aggregation (DLA) [3] model and for $\eta = 0$ the Eden model [4]. The significance of these models lies not only in their success as physical models for fractal features in a wide variety of phenomena, but also in their role as simple, well defined basic representatives of irreversible growth processes with infinite memory and long-range interactions [5-8]. These models can be studied in many different geometries, of which the most popular are the circular geometry [1, 3] in which growth starts in one point, and the cylinder geometry [6, 8]. The latter geometry consists of an $L \times M$ square lattice, with the two sides of height M identified so as to form a cylinder of circumference L. In the dielectric breakdown model in this geometry growth starts by putting the lower circle (the substrate) at electrostatic potential 0 and the upper one at potential 1. The cluster at time zero is the substrate and as in DBM in the circular geometry [1], one takes the probability of occupying one of its empty nearest neighbours, proportional to the modulus of the electric field there, to a power η (the growth parameter).

In the cylinder geometry one can distinguish two modes of growth [8-10]: a scaling regime for cluster heights (h) smaller than the cylinder circumference L, and the steady state for $h \gg L$. In the scaling regime many properties, like for instance the number of particles and the interface thickness, scale as a function of the height of the clusters. In the steady state, on the other hand, the different quantities, on average, become constant or scale trivially with the height.

Recently Pietronero *et al* [11] introduced a new theoretical approach for the determination of the fractal dimension of clusters grown with the dielectric breakdown

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model. This fixed scale transformation (FST) theory allows the computation of the box-counting dimension of one-dimensional intersections, perpendicular to the main growth direction, of clusters in the steady state in the cylinder geometry (see figure 1). In this letter we will present numerical results on these intersection set dimensions as a function of the height of the clusters, for six different values of the growth parameter η.

Our conclusions from the present numerical results are that, the box-counting dimension of intersection sets is a well defined quantity implying that these sets are (statistically) self-similar. In the scaling regime the intersection set dimensions depend on the height. This inhomogeneity is related to the self-affineness of DBM clusters and is discussed elsewhere [8]. The dependence then gives way to a constant intersection set dimension in the steady state. A comparison between our numerical results and those of the FST theory to third order in the closed-open approximation [11], gives good agreement. Since we numerically studied the dimensions in cylinders of different circumferences, we are also able to study finite-size effects. These turn out to be modest and quite the same for the different η values studied.

This letter is organised as follows. We will first discuss the box-counting procedure on the intersection sets and after some remarks about the clusters and the method of analysis, we will present results on the dimension of intersection sets as a function of their height in the cluster.

The set $S_h^{\perp}$ of occupied sites at height h shown in figure 1, belonging to a cluster grown in the cylinder geometry, is called an intersection set. In order to determine the box-counting dimension $D^{\perp}$ of this intersection set, we cover it with boxes of size $\varepsilon_n = 2^n$, $n = 0, 1, \ldots, k$, with $L = 2^k$. If the number of boxes of size ε_n containing at least one occupied site of the cluster is denoted by $N(\varepsilon_n)$, then $D^{\perp}$ is given by

$$N(\varepsilon_n) \sim \varepsilon_n^{D^{\perp}}. \tag{1}$$

Because of the translational invariance of the clusters in the main growth direction in the steady state, the box-counting dimension D_{st} is there given by

$$D_{\mathrm{st}} = 1 + D^{\perp}. \tag{2}$$

We will have numerical results on D_{st}, allowing us to verify the validity of (2) in a very direct manner.

In order to enhance the statistics in the numerical determination of the box-counting dimension $D^{\perp}(h)$ of the intersection set at height h, we rotated the clusters, while keeping the position of the boxes (intervals) fixed. In this way one obtains different boxings for the same cluster. Six rotations were made, which resulted from translations

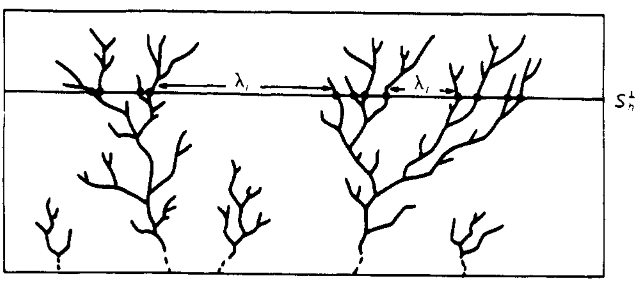

Figure 1. A cluster in a cylinder geometry with circumference L and an intersection set $S_h^{\perp}$ at height h.

of the cluster parallel to the substrate, over distances 1, 3, 5, 9, 17 and 33, modulo L. In addition, we also averaged over the number $n(\eta, L)$, of clusters grown with the growth parameter η in cylinders of circumference L. The value of $D^{\perp}(h)$ was determined from the slope of the plot of $\ln \varepsilon$ against the logarithm of the average number of boxes and versus the average of the logarithm of the number of boxes. Both methods gave the same results, within error bars.

Clusters were simulated on a CRAY X-MP [12] for four different cylinder circumferences, namely $L=256$, 128, 64 and 32, (i.e. $L=2^i$, $i=8,\ldots,5$) and six values of the growth parameter, $\eta=0.25$, 0.50, 0.75, 1.00, 1.25 and 2.00. The height of the cylinders was always taken to be three times their width. Their total number and the number of particles they contain each are shown in table 1. In figure 2 we show an example of an $\eta=0.75$ cluster in a cylinder of circumference $L=256$, containing 30 000 particles.

Table 1. The number of clusters $(n(\eta, L))$ of type (η, L) and the number of particles contained in each of them. For example, we simulated 20 clusters with growth parameter $\eta=1.00$ in a cylinder with circumference $L=128$, each containing 6000 particles.

η	$L=256$	$L=128$	$L=64$	$L=32$
0.25	10 × 30 000	20 × 20 000	20 × 5550	20 × 15 000
0.50	10 × 30 000	20 × 13 800	20 × 3900	20 × 990
0.75	10 × 30 000	20 × 8 400	20 × 3900	20 × 720
1.00	20 × 15 000	20 × 6 000	20 × 1500	20 × 540
1.25	20 × 15 000	20 × 3 000	20 × 1320	20 × 390
2.00	20 × 4 000	20 × 1 650	20 × 510	20 × 180

In figures 3(*a*) and 3(*b*), we show the dependence of $D^{\perp}(h)$ on h, for all η values in cylinders of respectively widths $L=256$ and $L=128$. The behaviour is qualitatively the same for all η values: we find that the fractal dimension of the intersection sets in the scaling regime varies with the height. It starts with the maximum dimension $D^{\perp}(0)=1$ at the lower electrode and then continuously decreases to the asymptotic value in the steady state. This shows that the *clusters in the scaling regime are not homogeneous fractals*. We show in [8, 13] that they are self-affine.

In principle the steady state goes on indefinitely, but due to limitations which are practical in nature, we only have a finite number of particles, which is the reason for the sudden drop at large heights. Note that neither the $\eta=0.25$ nor the $\eta=0.5$ clusters seem to have reached the steady state. The values of $D^{\perp}$ in the steady state, discussed below, should therefore be considered with some caution for these two values of the growth parameter: they may be too high.

In order to determine the average box-counting dimension $D^{\perp}_{st}$ of the steady state, the starting height h_{st} of the steady state and its length Δh, available from the present finite particle number simulations, was determined from the height interval in which the number of sites in the intersection sets was constant.

In figure 4 we show the plots of $\ln N(\varepsilon)$ against $\ln \varepsilon$ for $\eta=1$, $L=256$, 128, 64 and 32, involving an average over the height interval in the steady state. The dimension $D^{\perp}_{st}$ is given by minus the slopes of the straight lines through the points. These slopes were determined by a least-squares fit through all the points except for those corresponding with the largest and smallest boxes. The fact that these data lie on a straight line and thus that (1) holds, implies the self-similarity of the intersection sets. The

Figure 2. An $\eta = 0.75$ DBM cluster containing 30 000 particles, grown on the surface of a cylinder of circumference 256 and height 768. In the above picture there are periodic boundary conditions in the horizontal direction.

dimensions are shown in table 2. The $L = \infty$ values are the results of a $1/L \to 0$, linear extrapolation from the finite L values for $D^{\perp}$. This is achieved by means of a least-square fit through the finite $L = 2^5, \ldots, 2^8$ results, shown in figure 5. It should be noted that there is a systematic upward curvature in the $D^{\perp}$ against $1/L$ plots, implying that the $L = \infty$ values are somewhat larger than the linear estimates given in table 2. Except for the $\eta = 0.50$, the lack of pronounced finite-size effects nevertheless shows the results to be reliable and is another piece of numerical evidence for the reality of the η dependence of the fractal properties of DBM clusters. The strong curvature in the $\eta = 0.50$ data, is due to the already mentioned fact that the $L = 128$ and $L = 256$ clusters have not reached the steady state. The same is true for the $\eta = 0.25$ clusters, but it is less visible, because the dimension is very near to the upper bound 1.

Although the constancy of the value of $D^{\perp}$ in the steady state already guarantees the validity of (2), we performed an explicit check. In table 3 we show the numerical results for the box-counting dimension in the steady state of the $L = 128$ cylinder. These results are obtained by determining the box-counting dimension of an $L \times L$ part of the cluster in the steady state [8]. Using (2), we conclude that these results agree very well with those in table 2. This thus confirms the validity of this equation.

In figure 6 we plotted D_{st} using the values of $D^{\perp}(L = \infty)$ in table 2, together with the FST theory results for first order in the closed approximation and third order in the closed-open approximation. As was put forward in [11], the fact that the first-order approximation is too low compared with the numerical results is due to an overestimation of the screening due to the periodic boundary conditions on the small cells

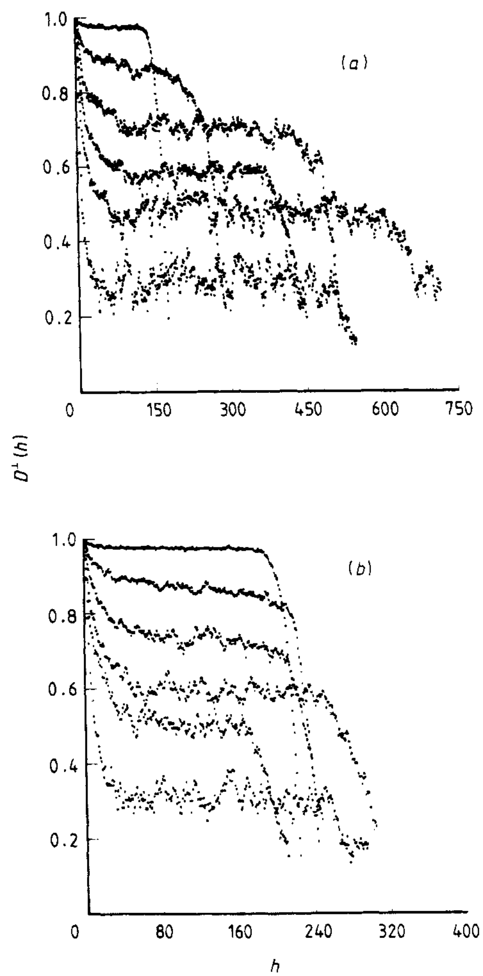

Figure 3. The dependence of the intersection set dimensions $D^{-}(h)$ as a function of the height h. From top to bottom the curves are for $\eta = 0.25$, 0.5, 0.75, 1.00, 1.25 and 2.00. (a) is for cylinder width $L = 256$ and (b) for $L = 128$.

considered and also due to the low order of the calculation. As was also already remarked in [11], the convergence for small η values is very slow in the FST approach. This is the reason for why also the closed-open approximation is too low for these values. On the other hand the convergence is fast for large η values, but we find that, for these values, the theoretical results overshoot the numerical result presented here. Now from the nature of this approximation, it follows (see [11] or [8]) that (after convergence) it is an upper bound for the full FST approach. We therefore believe that the results of the full FST approach will be closer to the numerical results.

We presented new numerical results on the dielectric breakdown model, mainly inspired by the fixed scale transformation theory [11]. In particular we concentrated on the box-counting dimension of intersection sets, for six different values of the growth parameter η and four cylinder circumferences, thereby allowing us to study finite-size effects. We find that in the scaling regime this dimension depends on the height of the intersections, but it then converges to a constant value in the steady state.

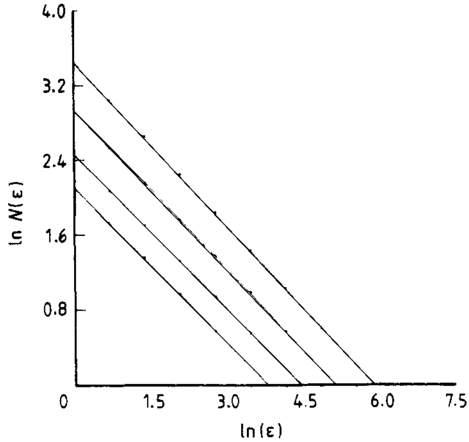

Figure 4. The log–log plots from which the box-counting dimension $D^{\perp}$ of the intersection sets were determined by means of a least-squares fit. The results, together with the values extrapolated to $L=\infty$, are shown in table 2.

Table 2. The box-counting dimension $D^{\perp}$ for the intersection sets in the steady state. The $L=\infty$ values are obtained from a linear extrapolation of the finite-size data against $1/L$.

η	$L=32$	$L=64$	$L=128$	$L=256$	$L=\infty$
0.25	0.983 ± 0.008	0.990 ± 0.005	0.973 ± 0.01	0.991 ± 0.004	0.985 ± 0.009
0.50	0.76 ± 0.02	0.77 ± 0.02	0.8 ± 0.02	0.86 ± 0.02	0.875 ± 0.03
0.75	0.648 ± 0.008	0.67 ± 0.02	0.676 ± 0.007	0.689 ± 0.006	0.691 ± 0.003
1.00	0.552 ± 0.005	0.549 ± 0.003	0.563 ± 0.04	0.588 ± 0.003	0.58 ± 0.01
1.25	0.455 ± 0.008	0.461 ± 0.004	0.473 ± 0.002	0.483 ± 0.004	0.489 ± 0.003
2.00	0.286 ± 0.003	0.278 ± 0.005	0.282 ± 0.005	0.300 ± 0.005	0.30 ± 0.01

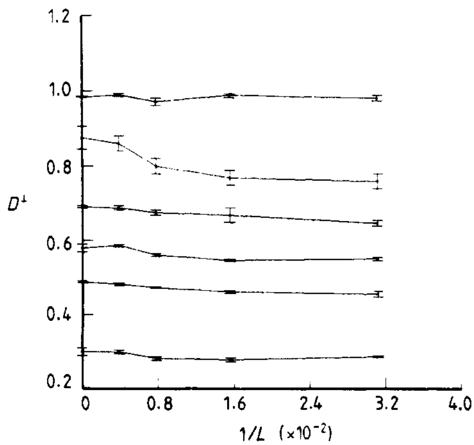

Figure 5. Linear extrapolation to $L=\infty$ of the box-counting dimension of the intersection sets in the steady state. The numbers are shown in table 2.

Table 3. The box-counting dimensions in the steady state of DBM clusters for different values of the growth parameter η in a cylinder of circumference $L = 128$.

η	D_u
0.25	1.977 ± 0.008
0.50	1.87 ± 0.02
0.75	1.73 ± 0.03
1.00	1.59 ± 0.01
1.25	1.50 ± 0.02
2.00	1.32 ± 0.01

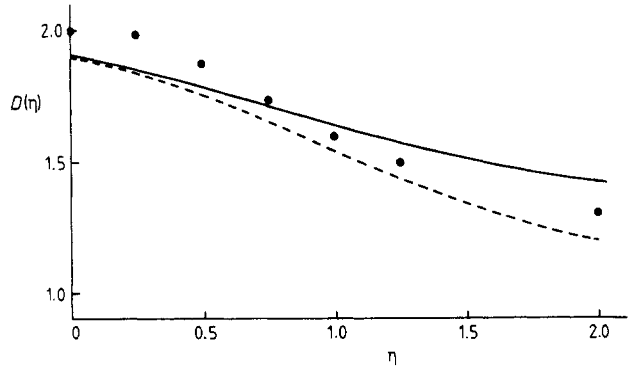

Figure 6. Dependence of the fractal dimension D on the growth parameter η for DBM in two dimensions. The broken curve refers to the simplest approximation to the full FST approach, in which one only considers cells with periodic boundary conditions up to first order. The full curve is the third-order result of the open-closed approximation to the full FST. This is the most sophisticated approximation in [11]. The dots present our numerical results.

These results show that, unlike in the scaling regime where they are self-affine, the clusters in the steady state are translational invariant in the main growth direction and self-similar in the perpendicular direction. A comparison between our findings and the theoretical results obtained from the FST approach, yields good agreement.

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